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ON THE EFFECTIVE MODULI OF ELASTIC COMPOSITE MATERIALS

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Abstract—A continuum theory for elastic composite materials consisting of a matrix and inclusions is presented. The approximate effective elastic moduli defined dynamically by means of the phase velocities of long plane harmonic waves are obtained for ellipsoid-, needle- and disk-shaped inclusions aligned or oriented at random. The results are compared with the effective moduli obtained by other methods.

1. INTRODUCTION

References[1, 2] contain a continuum theory developed for a laminated elastic material. The method, called "the effective stiffness theory", transforms a heterogeneous material into a homogeneous higher-order continuum with microstructure. The model shows the dispersion of harmonic waves. Although the dispersion curves are approximate, the phase velocities for long wave-lengths are exact[1]. In this case, the effective moduli defined dynamically by means of the phase velocities of long wave-lengths, are equal to the effective moduli defined statically. Later, the "effective stiffness" method was applied to unidirectional fibre-reinforced composites[3, 4] and to isotropic composites consisting of a matrix and spherical inclusions[5].

The material discussed in Ref. [4] was a fibre-reinforced composite with identical parallel fibres arranged in a hexagonal array. However, the same results apply to a material built up of composite elements consisting of fibres of circular cross-section surrounded by a cylindrical matrix jacket. The composite elements are arrayed in a way that makes the composite material macroscopically homogeneous and transversely isotropic. The radii of the composite elements can take arbitrarily small values so as to fill the space continuously. For such an arrangement of fibres, Hashin and Rosen[8] established exactly four of the five effective static moduli, and obtained the upper and the lower bound for the remaining fifth modulus. A comparison of these moduli with the effective dynamic moduli established by the application of the effective stiffness theory was presented in Ref. [4]. It was found that for a shear modulus of the fibres higher by a factor of 10 and 100 than the shear modulus of the matrix, and for Poisson's ratio of both fibres and matrix $\nu = 0.3$, every one of the four dynamic effective moduli was slightly higher than the exact value of the corresponding static effective modulus.

A continuous model of materials composed of a matrix and spherical inclusions was evolved in Ref. [5]. The composite element of this model was formed by a spherical inclusion and a spherical matrix jacket. The composite elements of various sizes were arranged to produce a macroscopically homogeneous and isotropic material. For this special microscopic arrangement, Hashin[6] obtained an exact static effective bulk modulus. In this case also, the approximate dynamic effective bulk modulus reported in Ref. [5] was slightly higher than those obtained in Ref. [6], the phase moduli having similar values as the moduli considered in Ref. [4].

Although, to the best of our knowledge, no general proof has so far been presented of the equivalence of the effective moduli defined statically (when the moduli bind between them the volume means of the components of the stress and strain tensor, with no considerations given to the inertia forces) and those defined dynamically (in terms of the phase velocities of long harmonic waves), the results reported above offer the possibility of using the effective stiffness method to calculate the approximate effective moduli. This problem is the subject of the present paper.

In Section 2, we consider the case of orthotropic ellipsoid-shaped inclusions. The material is assumed to be composed of ellipsoid-shaped composite elements. All ellipsoids are shaped alike and aligned. The sizes and the arrangement are such that the composite is macroscopically homogeneous and orthotropic. By examining the phase velocities of plane harmonic waves, nine approximate dynamic effective moduli are found for this case in explicit form. In Section 3, the model is generalized to ellipsoid-shaped inclusions of various shapes and orientations. Section 4 discusses an isotropic arrangement of inclusions of a shape of alike ellipsoids of revolution. Section 5 deals with needle- and disk-shaped inclusions. In Section 6 the results of illustrative calculations of the effective moduli are discussed and compared with the effective moduli obtained by other methods.

2. ALIGNED ELLIPSOID-SHAPE INCLUSIONS

Let the material consist of composite elements shown in Fig. 1. The inclusions are formed by ellipsoids. The local coordinate basis $\tilde{x}_i$ is placed at the centre of the inclusion so that the local axes $\tilde{x}_i$ lie in the principal directions of the ellipsoid. The lengths of the ellipsoid semi-axes are ar_1, br_1, cr_1, where a, b, c are positive dimensionless numbers defining the shape of the ellipsoid. The inclusion is surrounded by a matrix jacket whose outer surface is a similarly oriented ellipsoid with semi-axes ar_2, br_2, cr_2. r_1 and r_2 have the dimension of a length. The material consists of such composite elements of different sizes but oriented in the same way; hence all the local bases $\tilde{x}_i$ are parallel to the global basis x_i (Fig. 2). a, b, c are constant for all the elements. Although r_1 and r_2 vary, it is

$$\eta = \frac{r_1}{r_2} = \text{const.}, \quad \eta \neq 0, \eta \neq 1.$$

The sizes of the composite elements vary to infinitesimal values so that the space can be completely filled with them. The matrix is an elastic isotropic material with Lamé's constants λ_2, μ_2. The inclusions are of an elastic orthotropic material, the planes or orthotropy being the planes of symmetry of the inclusions. The distribution of the composite elements is assumed to be such as to make the composite material macroscopically homogeneous and orthotropic.

Consider now a single composite element (Fig. 1) with the x_{0i} coordinates of its centre. Introduce in it the local ellipsoidal coordinates r, φ, ϑ

$$\begin{aligned} \tilde{x}_1 &= ar \cos\varphi \sin\vartheta, \\ \tilde{x}_2 &= br \sin\varphi \sin\vartheta, \\ \tilde{x}_3 &= cr \cos\vartheta. \end{aligned} \tag{2.1}$$

Set forth the assumptions concerning the variations of the displacement vector in the composite element. The displacement vector $\tilde{u}_i^{(1)}$ in the inclusion is assumed to be linearly dependent on $\tilde{x}_i$, and the displacement vector $\tilde{u}_i^{(2)}$ in the matrix jacket of the element to be linearly dependent on r, i.e.

$$\begin{aligned} \tilde{u}_i^{(1)}(x_j, t) &= \tilde{u}_{0i}^{(1)}(x_{0j}, t) + \tilde{x}_j \tilde{\psi}_{ji}(x_{0j}, t), \\ \tilde{u}_i^{(2)}(x_j, t) &= \tilde{u}_{0i}^{(2)}(x_{0j}, r_2, \varphi, \vartheta, t) + (r - r_2)\tilde{U}_i(x_{0j}, \varphi, \vartheta, t). \end{aligned} \tag{2.2}$$

In the above, $\tilde{u}_{0i}^{(1)}(x_{0j}, t)$ denotes the displacement vector at the centre of the composite element, $\tilde{u}_{0i}^{(2)}(x_{0j}, r_2, \varphi, \vartheta, t)$ the displacement vector on the outer surface of the element at a point with the

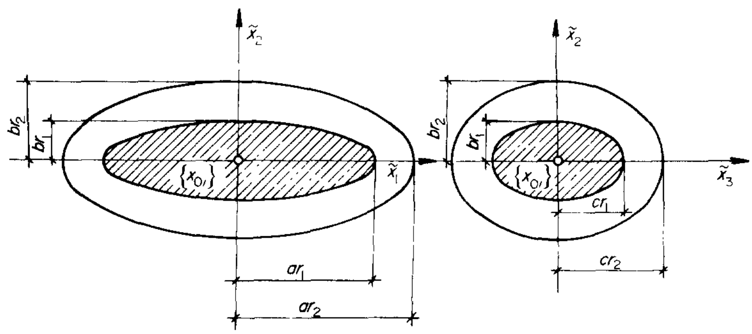

Fig. 1. Ellipsoidal composite element.

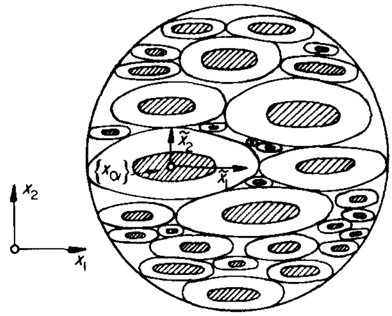

Fig. 2. Composite material with aligned ellipsoids.

local coordinates r_2, φ, ϑ. The wavy line denotes that we are dealing with components in the local basis $\tilde{x}_i$. $\tilde{u}_{0i}^{(1)}$, $\tilde{u}_{0i}^{(2)}$, $\tilde{\psi}_{ij}$, $\tilde{U}_i$ are functions defined only for discrete x_{0j}, r_2, i.e. at the centres or on the surfaces of the composite-elements. Since r_1 and r_2 are very small compared to the macroscopical unit length, we shall replace these functions by continuous functions defined for all x_j, $r > 0$. We shall further assume that $\tilde{u}_{0i}^{(1)}$ and $\tilde{u}_{0i}^{(2)}$ can be replaced by a single vector function $\tilde{u}_i$ called the gross-displacement, so that we obtain

$$\begin{aligned} \tilde{u}_{0i}^{(1)}(x_j, t) &= \tilde{u}_i(x_j, t), \\ \tilde{u}_{0i}^{(2)}(x_j, r_2, \varphi, \vartheta, t) &= \tilde{u}_i(x_j, t) + r_2[\tilde{u}_{i,1}(x_j, t)a \cos\varphi \sin\vartheta \\ &\quad + \tilde{u}_{i,2}(x_j, t)b \sin\varphi \sin\vartheta + \tilde{u}_{i,3}(x_j, t)c \cos\vartheta]. \end{aligned} \tag{2.3}$$

For $r = r_1$, the condition of continuity of displacement on the surface of the inclusion gives—with the use of (2.1) to (2.3)—the bond between $\tilde{U}_i$ and $\tilde{u}_i$, $\tilde{\psi}_{ij}$, viz.

$$\begin{aligned} (r_2 - r_1)\tilde{U}_i &= a \cos\varphi \sin\vartheta(r_2\tilde{u}_{i,1} - r_1\tilde{\psi}_{1i}) + b \sin\varphi \sin\vartheta(r_2\tilde{u}_{i,2} - r_1\tilde{\psi}_{2i}) \\ &\quad + c \cos\vartheta(r_2\tilde{u}_{i,3} - r_1\tilde{\psi}_{3i}). \end{aligned} \tag{2.4}$$

The bond between the neighbouring composite elements is guaranteed by the existence of $\tilde{u}_i$ and by the relations (2.3). Substitution of (2.4) into (2.2) yields

$$\begin{aligned} \tilde{u}_i^{(1)} &= \tilde{u}_i + \tilde{x}_j\tilde{\psi}_{ji} \quad \text{for} \quad 0 \le r \le r_1 \text{ (in the inclusion)}, \\ \tilde{u}_i^{(2)} &= \tilde{u}_i + \tilde{x}_j\tilde{H}_{ji}, \\ \tilde{H}_{ji} &= \frac{1}{r}\left[r_2\tilde{u}_{i,j} + \frac{r - r_2}{r_2 - r_1}(r_2\tilde{u}_{i,j} - r_1\tilde{\psi}_{ji})\right] \\ &\qquad \text{for} \quad r_1 \le r \le r_2 \text{ (in the matrix)}. \end{aligned} \tag{2.5}$$

The state of deformation in the medium is now described by $\tilde{u}_i$ and $\tilde{\psi}_{ij}$.

Denote the quantities referring to an ordinary composite element by the underlain index ξ. The strain energy $\underset{\xi}{W'}$ of the element ξ is defined by

$$\begin{aligned} \underset{\xi}{W'} &= \iiint\limits_{\underset{\xi}{V^{(1)}}} w'^{(1)}\, d\tilde{x}_1\, d\tilde{x}_2\, d\tilde{x}_3 + \iiint\limits_{\underset{\xi}{V^{(2)}}} w'^{(2)}\, d\tilde{x}_1\, d\tilde{x}_2\, d\tilde{x}_3, \\ w'^{(1)} &= \frac{1}{2}c_{11}\tilde{\epsilon}_{11}^{(1)2} + \frac{1}{2}c_{22}\tilde{\epsilon}_{22}^{(1)2} + \frac{1}{2}c_{33}\tilde{\epsilon}_{33}^{(1)2} + c_{12}\tilde{\epsilon}_{11}^{(1)}\tilde{\epsilon}_{22}^{(1)} + c_{13}\tilde{\epsilon}_{11}^{(1)}\tilde{\epsilon}_{33}^{(1)} \\ &\quad + c_{23}\tilde{\epsilon}_{22}^{(1)}\tilde{\epsilon}_{33}^{(1)} + 2c_{44}\tilde{\epsilon}_{23}^{(1)2} + 2c_{55}\tilde{\epsilon}_{13}^{(1)2} + 2c_{66}\tilde{\epsilon}_{12}^{(1)2}, \\ w'^{(2)} &= \frac{1}{2}\lambda_2\tilde{\epsilon}_{ii}^{(2)}\tilde{\epsilon}_{kk}^{(2)} + \mu_2\tilde{\epsilon}_{ij}^{(2)}\tilde{\epsilon}_{ij}^{(2)}, \\ \tilde{\epsilon}_{ij}^{(1)} &= \frac{1}{2}\left(\frac{\partial\tilde{u}_i^{(1)}}{\partial\tilde{x}_j} + \frac{\partial\tilde{u}_j^{(1)}}{\partial\tilde{x}_i}\right), \quad \tilde{\epsilon}_{ij}^{(2)} = \frac{1}{2}\left(\frac{\partial\tilde{u}_i^{(2)}}{\partial\tilde{x}_j} + \frac{\partial\tilde{u}_j^{(2)}}{\partial\tilde{x}_i}\right). \end{aligned} \tag{2.6}$$

In the above, $\underset{\xi}{V}^{(1)}$ and $\underset{\xi}{V}^{(2)}$ are, respectively, the volume of the inclusion and the volume of the matrix jacket. c_{11}, c_{12}, c_{13}, c_{22}, c_{23}, c_{33}, c_{44}, c_{55}, c_{66} are the material constants of an orthotropic inclusion with the constitutive equations

$$\begin{aligned}
\tilde{\tau}_{11}^{(1)} &= c_{11}\tilde{\epsilon}_{11}^{(1)} + c_{12}\tilde{\epsilon}_{22}^{(1)} + c_{13}\tilde{\epsilon}_{33}^{(1)},\\
\tilde{\tau}_{22}^{(1)} &= c_{12}\tilde{\epsilon}_{11}^{(1)} + c_{22}\tilde{\epsilon}_{22}^{(1)} + c_{23}\tilde{\epsilon}_{33}^{(1)},\\
\tilde{\tau}_{33}^{(1)} &= c_{13}\tilde{\epsilon}_{11}^{(1)} + c_{23}\tilde{\epsilon}_{22}^{(1)} + c_{33}\tilde{\epsilon}_{33}^{(1)},\\
\tilde{\tau}_{23}^{(1)} = 2c_{44}\tilde{\epsilon}_{23}^{(1)}, \quad & \tilde{\tau}_{13}^{(1)} = 2c_{55}\tilde{\epsilon}_{13}^{(1)}, \quad \tilde{\tau}_{12}^{(1)} = 2c_{66}\tilde{\epsilon}_{12}^{(1)},
\end{aligned} \tag{2.7}$$

where $\tilde{\tau}_{ij}^{(1)}$ denotes the stress tensor in the inclusion. By using (2.5) we obtain the strain energy $\underset{\xi}{W}$ per unit volume of the composite element ξ

$$\underset{\xi}{W} = \frac{\underset{\xi}{W'}}{\underset{\xi}{V}^{(1)} + \underset{\xi}{V}^{(2)}}$$

in the form

$$\underset{\xi}{W} = \frac{1}{2}\tilde{A}_{ijkl}\tilde{\epsilon}_{ij}\tilde{\epsilon}_{kl} + \tilde{B}_{ijkl}\tilde{\epsilon}_{ij}\tilde{\gamma}_{kl} + \frac{1}{2}\tilde{C}_{ijkl}\tilde{\gamma}_{ij}\tilde{\gamma}_{kl}, \tag{2.8}$$

where

$$\tilde{\epsilon}_{ij} = \tilde{u}_{(i,j)}, \qquad \tilde{\gamma}_{ij} = \tilde{u}_{j,i} - \tilde{\psi}_{ij}.$$

The material tensors $\tilde{A}_{ijkl}$, $\tilde{B}_{ijkl}$, $\tilde{C}_{ijkl}$ are given in the Appendix, eqn (A1). If W denotes the strain energy per unit volume of the composite material, then

$$W = \frac{\sum_{\xi} \underset{\xi}{W}(\underset{\xi}{V}^{(1)} + \underset{\xi}{V}^{(2)})}{\sum_{\xi} (\underset{\xi}{V}^{(1)} + \underset{\xi}{V}^{(2)})}, \tag{2.9}$$

where the summation is taken over all the composite elements ξ contained in a small macroscopical volume.

The kinetic energy $\underset{\xi}{K'}$ of the composite element ξ is defined by

$$\underset{\xi}{K'} = \frac{1}{2}\iiint_{\underset{\xi}{V}^{(1)}} \rho_1 \dot{\tilde{u}}_i^{(1)}\dot{\tilde{u}}_i^{(1)}\,\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2\,\mathrm{d}\tilde{x}_3 + \frac{1}{2}\iiint_{\underset{\xi}{V}^{(2)}} \rho_2 \dot{\tilde{u}}_i^{(2)}\dot{\tilde{u}}_i^{(2)}\,\mathrm{d}\tilde{x}_1\,\mathrm{d}\tilde{x}_2\,\mathrm{d}\tilde{x}_3 \tag{2.10}$$

where ρ_1 and ρ_2 are, respectively, the mass density of the inclusions and the mass density of the matrix. The dot above a quantity denotes the derivative with respect to time. $\underset{\xi}{K'}$ is calculated from (2.10) by using (2.5). Since only the phase velocities of very long harmonic waves will be required later on, in the present case we can neglect in $\underset{\xi}{K'}$ all the terms producing dispersion. Without these terms the kinetic energy $\underset{\xi}{K}$ of unit volume of the composite element will turn out to be

$$\underset{\xi}{K} = \frac{1}{2}\bar{\rho}\dot{\tilde{u}}_i\dot{\tilde{u}}_i, \qquad \bar{\rho} = \eta^3\rho_1 + (1-\eta^3)\rho_2. \tag{2.11}$$

Analogously to (2.9) we define the kinetic energy K of unit volume of the composite material.

Since $\underset{\xi}{W}$, $\underset{\xi}{K}$ depend on $\underset{\xi}{r_1}$, $\underset{\xi}{r_2}$ only through the intermediary of constant η, $\underset{\xi}{W}$ in (2.9) can be factored out of the summation (similarly as $\underset{\xi}{K}$), and we obtain

$$W = \underset{\xi}{W}, \qquad K = \underset{\xi}{K}.$$

Since in this case, the basis $\tilde{x}_i$ is parallel to the basis x_i for all ξ, the wavy line may be left out.

Let V denote a fixed regular region, and t_1, t_2 fixed times. For independent variations δu_i, $\delta\psi_{ij}$ for which

$$\delta u_i = \delta\psi_{ij} = 0$$

on the surface S of the region V, Hamilton's principle is of the form

$$\delta\int_{t_1}^{t_2}\int_V (K - W)\,\mathrm{d}V\,\mathrm{d}t = 0. \tag{2.12}$$

The sought equations of motion are Euler's conditions of the variational principle (2.12)

$$\tau_{ij,i} + \sigma_{ij,i} - \bar{\rho}\ddot{u}_j = 0, \qquad \sigma_{ij} = 0 \tag{2.13}$$

where

$$\tau_{ij} = \frac{\partial W}{\partial \epsilon_{ij}}, \qquad \sigma_{ij} = \frac{\partial W}{\partial \gamma_{ij}}. \tag{2.14}$$

By substituting (2.8) into (2.14), and (2.14) into (2.13) we obtain 12 equations of motion expressed in terms of u_i, ψ_{ij}.

A plane harmonic wave propagating in the composite material is of the form

$$u_l = U_l\,\mathrm{e}^{ik(n_jx_j - ct)}, \qquad \psi_{lm} = \Psi_{lm}\,\mathrm{e}^{ik(n_jx_j - ct)}. \tag{2.15}$$

Here U_l, Ψ_{lm} are the constant amplitudes, k is the wave number, c the phase velocity and n_i is the unit vector defining the direction of propagation. After substituting (2.15) into (2.13) expressed in terms of u_i, ψ_{ij}, we obtain a homogeneous system of equations for U_l, ψ_{lm}. The condition of non-zero amplitudes is that the determinant of the system should be zero. This is the condition which enables the wave to propagate, and from this condition we can obtain the phase velocity c of the wave. Since the dispersive terms in (2.10) were neglected, c is independent of k, i.e. c is independent of the wave length. Generally, if dispersion were considered, this c would correspond to the phase velocity of infinitely long waves only. In a homogeneous orthotropic material with the material constants

$$\bar{c}_{11}, \bar{c}_{12}, \bar{c}_{13}, \bar{c}_{22}, \bar{c}_{23}, \bar{c}_{33}, \bar{c}_{44}, \bar{c}_{55}, \bar{c}_{66} \tag{2.16}$$

the meaning of which is similar as in (2.7) there is no dispersion and c are constant. The moduli (2.16) determined from the condition that the phase velocities of harmonic waves in a homogeneous material are the same as the phase velocities of the corresponding waves in the composite material for infinite wave lengths, will be termed the dynamic effective moduli. This comparison yields the following final results (for details see the Appendix, eqns (A2)–(A4)).

$$\bar{c}_{11} = \frac{1}{C}\begin{vmatrix} \tilde{A}_{1111} & \tilde{B}_{1111} & \tilde{B}_{1122} & \tilde{B}_{1133} \\ \tilde{B}_{1111} & \tilde{C}_{1111} & \tilde{C}_{1122} & \tilde{C}_{1133} \\ \tilde{B}_{1122} & \tilde{C}_{1122} & \tilde{C}_{2222} & \tilde{C}_{2233} \\ \tilde{B}_{1133} & \tilde{C}_{1133} & \tilde{C}_{2233} & \tilde{C}_{3333} \end{vmatrix}, \qquad \bar{c}_{12} = \frac{1}{C}\begin{vmatrix} \tilde{A}_{1122} & \tilde{B}_{2211} & \tilde{B}_{2222} & \tilde{B}_{2233} \\ \tilde{B}_{1111} & \tilde{C}_{1111} & \tilde{C}_{1122} & \tilde{C}_{1133} \\ \tilde{B}_{1122} & \tilde{C}_{1122} & \tilde{C}_{2222} & \tilde{C}_{2233} \\ \tilde{B}_{1133} & \tilde{C}_{1133} & \tilde{C}_{2233} & \tilde{C}_{3333} \end{vmatrix},$$

$$\bar{c}_{22}=\frac{1}{C}\begin{vmatrix}\tilde{A}_{2222} & \tilde{B}_{2211} & \tilde{B}_{2222} & \tilde{B}_{2233}\\ \tilde{B}_{2211} & \tilde{C}_{1111} & \tilde{C}_{1122} & \tilde{C}_{1133}\\ \tilde{B}_{2222} & \tilde{C}_{1122} & \tilde{C}_{2222} & \tilde{C}_{2233}\\ \tilde{B}_{2233} & \tilde{C}_{1133} & \tilde{C}_{2233} & \tilde{C}_{3333}\end{vmatrix}, \qquad \bar{c}_{23}=\frac{1}{C}\begin{vmatrix}\tilde{A}_{2233} & \tilde{B}_{3311} & \tilde{B}_{3322} & \tilde{B}_{3333}\\ \tilde{B}_{2211} & \tilde{C}_{1111} & \tilde{C}_{1122} & \tilde{C}_{1133}\\ \tilde{B}_{2222} & \tilde{C}_{1122} & \tilde{C}_{2222} & \tilde{C}_{2233}\\ \tilde{B}_{2233} & \tilde{C}_{1133} & \tilde{C}_{2233} & \tilde{C}_{3333}\end{vmatrix}$$

$$\bar{c}_{33}=\frac{1}{C}\begin{vmatrix}\tilde{A}_{3333} & \tilde{B}_{3311} & \tilde{B}_{3322} & \tilde{B}_{3333}\\ \tilde{B}_{3311} & \tilde{C}_{1111} & \tilde{C}_{1122} & \tilde{C}_{1133}\\ \tilde{B}_{3322} & \tilde{C}_{1122} & \tilde{C}_{2222} & \tilde{C}_{2233}\\ \tilde{B}_{3333} & \tilde{C}_{1133} & \tilde{C}_{2233} & \tilde{C}_{3333}\end{vmatrix}, \qquad \bar{c}_{13}=\frac{1}{C}\begin{vmatrix}\tilde{A}_{1133} & \tilde{B}_{1111} & \tilde{B}_{1122} & \tilde{B}_{1133}\\ \tilde{B}_{3311} & \tilde{C}_{1111} & \tilde{C}_{1122} & \tilde{C}_{1133}\\ \tilde{B}_{3322} & \tilde{C}_{1122} & \tilde{C}_{2222} & \tilde{C}_{2233}\\ \tilde{B}_{3333} & \tilde{C}_{1133} & \tilde{C}_{2233} & \tilde{C}_{3333}\end{vmatrix} \tag{2.17}$$

$$\bar{c}_{44}=\frac{\begin{vmatrix}\tilde{A}_{2323} & \tilde{B}_{2323} & \tilde{B}_{2332}\\ \tilde{B}_{2323} & \tilde{C}_{2323} & \tilde{C}_{2332}\\ \tilde{B}_{2332} & \tilde{C}_{2332} & \tilde{C}_{3232}\end{vmatrix}}{\begin{vmatrix}\tilde{C}_{2323} & \tilde{C}_{2332}\\ \tilde{C}_{2332} & \tilde{C}_{3232}\end{vmatrix}}, \qquad \bar{c}_{55}=\frac{\begin{vmatrix}\tilde{A}_{1313} & \tilde{B}_{1313} & \tilde{B}_{1331}\\ \tilde{B}_{1313} & \tilde{C}_{1313} & \tilde{C}_{1331}\\ \tilde{B}_{1331} & \tilde{C}_{1331} & \tilde{C}_{3131}\end{vmatrix}}{\begin{vmatrix}\tilde{C}_{1313} & \tilde{C}_{1331}\\ \tilde{C}_{1331} & \tilde{C}_{3131}\end{vmatrix}},$$

$$\bar{c}_{66}=\frac{\begin{vmatrix}A_{1212} & \tilde{B}_{1212} & \tilde{B}_{1221}\\ \tilde{B}_{1212} & \tilde{C}_{1212} & \tilde{C}_{1221}\\ \tilde{B}_{1221} & \tilde{C}_{1221} & \tilde{C}_{2121}\end{vmatrix}}{\begin{vmatrix}\tilde{C}_{1212} & \tilde{C}_{1221}\\ \tilde{C}_{1221} & \tilde{C}_{2121}\end{vmatrix}}, \qquad \text{with} \quad C=\begin{vmatrix}\tilde{C}_{1111} & \tilde{C}_{1122} & \tilde{C}_{1133}\\ \tilde{C}_{1122} & \tilde{C}_{2222} & \tilde{C}_{2233}\\ \tilde{C}_{1133} & \tilde{C}_{2233} & \tilde{C}_{3333}\end{vmatrix}.$$

3. ELLIPSOIDS OF DIFFERENT SHAPE AND ORIENTATION

The inclusions considered in the preceding section, were all aligned, similar ellipsoids. Imagine now, that a unit volume of the composite contains a large quantity of composite ellipsoids having different shape, size and orientation, and that η is variable also. It is only assumed that the arrangement of the inclusions is special in the sense that for a given inclusion the surface of the composite element and the surface of the inclusion are similar ellipsoids. Consider the material of the inclusion to be orthotropic and that of the matrix isotropic.

Use the notation

$$\alpha\equiv\frac{a}{c}, \qquad \beta\equiv\frac{b}{c}$$

and let there be for each composite element $\alpha \in (\alpha_1, \alpha_2)$, $\beta \in (\beta_1, \beta_2)$, where $\alpha_1<\alpha_2$, $\beta_1<\beta_2$ are fixed positive finite numbers; further, let there be $\eta \in (0, 1)$, $\varphi \in \langle 0, 2\pi)$, $\vartheta \in \langle 0, (\pi/2)\rangle$, $\omega \in \langle 0, \pi)$, where φ, ϑ, ω are Euler's angles which relate the basix $\bar{x}_i$ and the basis x_i (see the Appendix). If we know the volume rate of composite elements in a macroscopical unit of the composite for various orientations of the bases $\bar{x}_i$, for various η, α, β, i.e. if we know the function $F(\varphi, \vartheta, \omega, \eta, \alpha, \beta)$ satisfying the condition

$$\int_{\beta_1}^{\beta_2}\int_{\alpha_1}^{\alpha_2}\int_0^1\int_0^{\pi}\int_0^{2\pi}\int_0^{\pi/2} F(\varphi, \vartheta, \omega, \eta, \alpha, \beta)\sin\vartheta \, d\vartheta \, d\varphi \, d\omega \, d\eta \, d\alpha \, d\beta = 1,$$

we can define the strain energy $\mathscr{W}$ of unit volume of this composite material by the relation

$$\mathscr{W}=\int_{\beta_1}^{\beta_2}\int_{\alpha_1}^{\alpha_2}\int_0^1\int_0^{\pi}\int_0^{2\pi}\int_0^{\pi/2} F(\varphi, \vartheta, \omega, \eta, \alpha, \beta)W(\varphi, \vartheta, \omega, \eta, \alpha, \beta)\sin\vartheta \, d\vartheta \, d\varphi \, d\omega \, d\eta \, d\alpha \, d\beta. \tag{3.1}$$

$W(\varphi, \vartheta, \omega, \eta, \alpha, \beta)$ of (3.1) is given by the expression Win (2.8) in which the index ξ denotes that W refers to the composite elements with the parameters φ, ϑ, ω, η, α, β. If we carry out in (2.8) the transformation $\tilde{\epsilon}_{ij}$, $\tilde{\gamma}_{ij}$ from the basis $\tilde{x}_i$ to the basis x_i using the matrix T_{ij} (see (A5), (A6)), we obtain for $\mathscr{W}$ the equation

$$\mathscr{W} = \frac{1}{2} A_{ijkl}\epsilon_{ij}\epsilon_{kl} + B_{ijkl}\epsilon_{ij}\gamma_{kl} + \frac{1}{2} C_{ijkl}\gamma_{ij}\gamma_{kl}, \tag{3.2}$$

where

$$A_{ijkl} = \int_{\beta_1}^{\beta_2}\int_{\alpha_1}^{\alpha_2}\int_0^1\int_0^{\pi}\int_0^{2\pi}\int_0^{\pi/2} F(\varphi, \vartheta, \omega, \eta, \alpha, \beta)\tilde{A}_{mnop}(\eta, \alpha, \beta)T_{im}T_{jn}T_{ko}T_{lp} \times \sin\vartheta\, d\vartheta\, d\varphi\, d\omega\, d\eta\, d\alpha\, d\beta$$

and similarly for B_{ijkl}, C_{ijkl}. $\tilde{A}_{mnop}$, $\tilde{B}_{mnop}$, $\tilde{C}_{mnop}$ are given in (A1). T_{ij} in (3.2) depend on Euler's angles φ, ϑ, ω and are given by (A5), (A6).

The kinetic energy density $\mathscr{K}$ could be defined and the equations of motion derived using a procedure analogous to that of Section 2. According to the form of the function F, a comparison with a homogeneous material displaying the same material symmetries enables us to seek the approximate effective moduli by identifying the phase velocities of the corresponding harmonic waves.

4. INCLUSIONS IN THE FORM OF ISOTROPICALLY DISTRIBUTED ELLIPSOIDS OF REVOLUTION

In this section we shall examine a composite material the composite elements of which contain inclusions in the form of ellipsoids of revolution of identical shape and various sizes at a constant η, oriented and arranged so as to, make the composite macroscopically homogeneous and isotropic. Let the inclusions be transversely isotropic with the isotropy axis in the axis of revolution $\tilde{x}_3$ so that for an isotropic matrix, each composite element is symmetrical, both geometrically and materially, about $\tilde{x}_3$. Transverse isotropy is a special case of orthotropy for

$$c_{11} = c_{22}, \quad c_{44} = c_{55}, \quad c_{13} = c_{23}, \quad 2c_{66} = c_{11} - c_{12}, \tag{4.1}$$

hence the material of the inclusions is defined by five constants, c_{11}, c_{12}, c_{13}, c_{33}, c_{44}. Further, according to the assumptions,

$$\eta = \text{const.}, \quad a = b, \quad \alpha = \text{const.}$$

We are dealing with a special case of the material discussed in Section 3. Because of the axial symmetry of the elements about $\tilde{x}_3$, we set $\omega = 0$ in (A5). As a result, (3.2) simplifies to

$$\mathscr{W} = \frac{1}{2} A_{ijkl}\epsilon_{ij}\epsilon_{kl} + B_{ijkl}\epsilon_{ij}\gamma_{kl} + \frac{1}{2} C_{ijkl}\gamma_{ij}\gamma_{kl}, \tag{4.2}$$

where

$$A_{ijkl} = \frac{1}{2\pi}\int_0^{\pi/2}\int_0^{2\pi} \tilde{A}_{mnop}T_{im}T_{jn}T_{ko}T_{lp}\sin\vartheta\, d\varphi\, d\vartheta$$

and similarly for B_{ijkl}, C_{ijkl}, T_{ij} in (4.2) being of the form of (A7). $\tilde{A}_{mnop}$ in (4.2), and similarly $\tilde{B}_{mnop}$, $\tilde{C}_{mnop}$ are obtained by substituting (4.1) into (A1). Evaluation of the integral in (4.2) gives A_{ijkl}, B_{ijkl}, C_{ijkl}. Their non-zero components are of the form of (A8). The kinetic energy density $\mathscr{K}$ continues to be

$$\mathscr{K} = \frac{1}{2}\bar{\rho}\dot{u}_i\dot{u}_i, \quad \bar{\rho} = \eta^3\rho_1 + (1-\eta^3)\rho_2.$$

Similarly as in Section 2, the equations of motion can be obtained by the application of Hamilton's principle. Comparing the phase velocities of the longitudinal and transverse wave for the model with microstructure with those for a classical homogeneous isotropic continuum, the effective Lamé's moduli $\bar{\lambda}$, $\bar{\mu}$ become

$$\bar{\mu} = b_6 + \frac{2b_7b_8b_{10} - b_9(b_7^2 + b_8^2)}{(b_9 - b_{10})(b_9 + b_{10})},$$

$$\bar{\lambda}+2\bar{\mu} = b_1 + \frac{4b_2b_3b_5 - 2b_3^2b_4 - b_2^2(b_4+b_5)}{(b_4-b_5)(b_4+2b_5)}. \tag{4.3}$$

$b_1, b_2, \ldots, b_{10}$ are given in (A9).

If the material of the inclusions is isotropic, we have

$$c_{11} = c_{33} = \lambda_1 + 2\mu_1, \quad c_{12} = c_{13} = \lambda_1, \quad c_{44} = \mu_1. \tag{4.4}$$

λ_1, μ_1 are Lamé's constants of the material of the inclusions. For this particular case we obtain from (4.3)

$$\begin{aligned} \bar{\mu} &= \left\langle 1 + \frac{A(b_3'+b_7')}{b_3'+b_7'+2A} \right\rangle \mu_2, \\ \bar{\lambda}+2\bar{\mu} &= \left\langle \delta_2 + \frac{B(b_2'-b_3')(b_2'+2b_3')+2Ab_2'(3B-4A)}{(b_2'-b_3'+2A)(b_2'+2b_3'+3B-4A)} \right\rangle \mu_2. \end{aligned} \tag{4.5}$$

In (4.5) we introduced the notation

$$\begin{aligned} &A = \eta^3(\gamma-1), \quad B = \eta^3(\delta_1\gamma - \delta_2), \\ &\gamma = \frac{\mu_1}{\mu_2}, \quad \delta_\epsilon = \frac{2(1-\nu_\epsilon)}{1-2\nu_\epsilon}, \quad \nu_\epsilon = \frac{\lambda_\epsilon}{2(\lambda_\epsilon+\mu_\epsilon)}, \epsilon = 1, 2; \\ &b_2' = \frac{1}{\mu_2} b_2 = \frac{V}{15}\left\langle (41\delta_2+14) + 2(\delta_2+4)\left(\alpha^2 + \frac{1}{\alpha^2}\right)\right\rangle, \\ &b_3' = \frac{1}{\mu_2} b_3 = \frac{V}{15}(\delta_2 - 1)\left\langle 17 - \left(\alpha^2 + \frac{1}{\alpha^2}\right)\right\rangle, \\ &b_7' = \frac{1}{\mu_2} b_7 = \frac{V}{15}\left\langle (7\delta_2+48) + 2(2\delta_2+3)\left(\alpha^2 + \frac{1}{\alpha^2}\right)\right\rangle. \end{aligned} \tag{4.6}$$

V is defined in (A1). The case of spherical inclusions is obtained on setting $\alpha = 1$ in (4.5), (4.6). The effective moduli turn out to be

$$\begin{aligned} \bar{\mu} &= \left\langle 1 + \frac{V\eta^3(\gamma-1)(2\delta_2+3)}{2\eta^3(\gamma-1)+V(2\delta_2+3)} \right\rangle \mu_2, \\ \bar{\kappa} &= \left\langle 1 + \frac{5\eta^3 V\delta_2[3(\delta_1\gamma-\delta_2)-4(\gamma-1)]}{(3\delta_2-4)\{5V\delta_2+\eta^3[3(\delta_1\gamma-\delta_2)-4(\gamma-1)]\}} \right\rangle \kappa_2. \end{aligned} \tag{4.7}$$

κ_2 is the bulk modulus of the matrix, i.e.

$$\kappa_2 = \lambda_2 + \frac{2}{3}\mu_2.$$

Similarly, $\bar{\kappa}$ denotes the approximate effective bulk modulus of the composite. The case of spherical inclusions, including dispersion, was examined in Ref. [5].

5. RANDOMLY DISTRIBUTED NEEDLE- AND DISK-SHAPED INCLUSIONS

For inclusions in the shape of ellipsoids of revolution, examined in Section 4, the effective moduli at $\alpha \to 0$ are the same as those at $\alpha \to \infty$. The first case represents highly flattened, the second case highly elongated ellipsoids. As will be shown in Section 6, the model discussed in Section 4 for composite elements shaped like ellipsoids of revolution, is not suitable for needle-shaped and disk-shaped inclusions. The reason is that with α tending away from 1, the effective moduli (4.5) at $\gamma > 1$ grow larger, and in the limit $\alpha \to \infty$ ($\alpha \to 0$) exceed the upper bound of Hashin–Shtrikman[7].

A model better suited for materials with needle-shaped inclusions is that of fibre-reinforced composites derived in [4]. As shown in Fig. 3, the composite element is a long cylinder of radius r_2, in the $\bar{x}_3$-axis of which is a fibre of circular cross-section with radius r_1. r_1 and r_2 are varied at

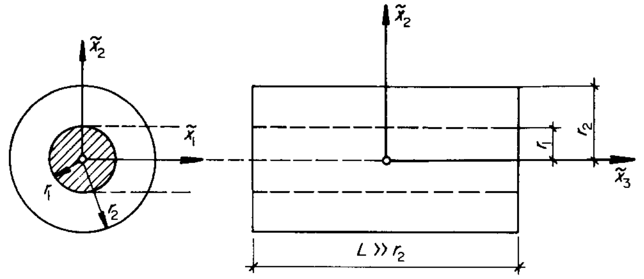

Fig. 3. Needle-shaped composite element.

constant $\eta = r_1/r_2$ and the composite elements are oriented and arranged so as to fill the space and make the composite macroscopically homogeneous and isotropic. The strain energy and the kinetic energy of such composite elements were calculated in[4]. Proceeding as outlined in Section 4, for isotropic needles the effective moduli are obtained in the form of (4.3), with

$$\begin{aligned}
&\frac{1}{\mu_2} b_1 = 2A' + 3B' + V'(3\delta_2 + 2) + \delta_2, \quad \frac{1}{\mu_2} b_2 = 2A' - 2B' + V'(3\delta_2 + 2),\\
&\frac{1}{\mu_2} b_3 = 9A' - 4B' + V'(\delta_2 - 1), \quad \frac{1}{\mu_2} b_4 = 2A' + 8B' + V'(3\delta_2 + 2),\\
&\frac{1}{\mu_2} b_5 = -11A' + 6B' + V'(\delta_2 - 1), \quad \frac{1}{\mu_2} b_6 = 4A' + B' + V'(\delta_2 + 4) + 1,\\
&\frac{1}{\mu_2} b_7 = -A' + B' + V'(\delta_2 + 4), \quad \frac{1}{\mu_2} b_8 = -6A' + B' + V'(\delta_2 - 1),\\
&\frac{1}{\mu_2} b_9 = 9A' + B' + V'(\delta_2 + 4), \quad \frac{1}{\mu_2} b_{10} = 4A' + B' + V'(\delta_2 - 1).
\end{aligned} \tag{5.1}$$

In (5.1) we introduced the notation

$$A' = \frac{1}{15}\eta^2(\gamma - 1), \quad B' = \frac{1}{15}\eta^2(\delta_1\gamma - \delta_2), \quad V' = -\frac{2}{15}\frac{\eta^2}{(1-\eta)^2}\lg \eta.$$

For disk-shaped inclusions we shall use the model discussed in[1, 2]. The composite element (see Fig. 4) is shaped like a disk whose radius R is considerably larger than the thickness $2r_2$. The isotropic inclusion has a thickness $2r_1$. Again, r_1 and r_2 are varied at constant $\eta = r_1/r_2$ and the composite elements are oriented and arranged so as to fill the space and make the composite macroscopically homogeneous and isotropic. The strain energy and the kinetic energy of such

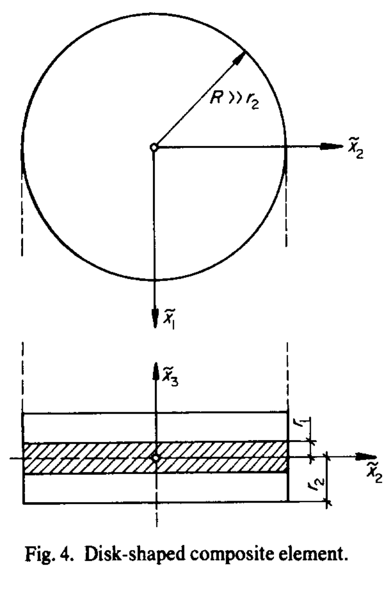

Fig. 4. Disk-shaped composite element.

composite elements were calculated in[2]. The effective moduli for isotropic inclusions turn out to be

$$\bar{\mu} = \left\langle 1+3A''-\frac{2A''^2}{d_2+d_3}\right\rangle \mu_2,$$

$$\bar{\lambda}+2\bar{\mu} = \left\langle \delta_2+3B''+\frac{B''(8A''-3B'')(d_1-d_2)-8A''^2 d_1}{(d_1-d_2)(d_1+2d_2)}\right\rangle \mu_2 \tag{5.2}$$

with

$$d_1=\frac{\eta}{15}\left[\gamma(3\delta_1+2)+\frac{\eta}{1-\eta}(3\delta_2+2)\right],\quad d_2=\frac{\eta}{15}\left[\gamma(\delta_1-1)+\frac{\eta}{1-\eta}(\delta_2-1)\right],$$

$$d_3=\frac{\eta}{15}\left[\gamma(\delta_1+4)+\frac{\eta}{1-\eta}(\delta_2+4)\right],\quad A''=\frac{\eta}{3}(\gamma-1),\quad B''=\frac{\eta}{3}(\delta_1\gamma-\delta_2).$$

γ, δ_1, δ_2 were defined in (4.6).

6. NUMERICAL RESULTS

The greater is the difference between the phase moduli, the broader are the bounds[6–8], between which must lie the effective moduli for given volume fractions of the inclusions $\bar{\eta}$, i.e. the greater is the scatter of the effective moduli in dependence on the shape and arrangement of the inclusions. We shall make a comparison for the case of very hard inclusions, i.e. we choose

$$\gamma=\frac{\mu_1}{\mu_2}=100,\quad \nu_1=\nu_2=0.3.$$

Some results for unidirectional, fibre-reinforced composites are shown in Table 1. The approximate dynamic effective moduli given by (5.5) in[4] are compared with four exact static effective moduli from[8]. The approximate effective moduli are slightly higher than the exact values.

The results for the bulk modulus $\bar{\kappa}$ of macroscopically isotropic composites with isotropic phases are given in Table 2. For spherical inclusions $\bar{\kappa}$ calculated from (4.7) are compared with the exact values given by (38) in[6]. As to $\bar{\kappa}$ for needle- and disk-shaped inclusions, we compare (4.3), (5.1) and (5.2) with the results of[13] whose authors, Christensen and Waals, considered a macroscopically isotropic arrangement of needle-shaped inclusions. They used five moduli for parallel-oriented fibres and integrated over all fibre directions. Four of the five moduli are known exactly, while the bounds of variation are known for the fifth (the transverse shear) modulus[8]. For randomly oriented fibres it is found that the static bulk modulus $\bar{\kappa}'$ is not affected by the choice of the transverse shear modulus. The same can be done for disk-shaped inclusions. We use the five moduli for parallel-oriented disks as those of the laminated medium (which are all

Table 1.

	(5.5) in[4]				Hashin–Rosen[8]			
$\bar{\eta}$	C_{33}/μ_2	C_{44}/μ_2	C_{13}/μ_2	$\frac{1}{2}(C_{11}+C_{12})/\mu_2$	C_{33}/μ_2	C_{44}/μ_2	C_{13}/μ_2	$\frac{1}{2}(C_{11}+C_{12})/\mu_2$
0.3	81.265	1.857	2.409	4.015	81.249	1.833	2.382	3.970
0.5	133.437	2.940	3.562	5.937	133.425	2.922	3.542	5.904
0.7	186.494	5.382	6.192	10.320	186.488	5.372	6.180	10.299

Table 2.

$\bar{\eta}$	$\bar{\kappa}/\mu_2$ spheres (4.7)	$\bar{\kappa}/\mu_2$ needles (4.3), (5.1)	$\bar{\kappa}/\mu_2$ disks (5.2)	$\bar{\kappa}/\mu_2$ spheres (33) in[6]	$\bar{\kappa}'/\mu_2$ needles [13]	$\bar{\kappa}'/\mu_2$ disks [13]
0.3	3.71	11.86	27.49	3.65	11.85	27.33
0.5	5.61	19.07	44.33	5.56	19.02	44.24
0.7	9.94	28.06	62.40	9.91	28.04	62.38

known exactly), and similarly as in[13] integrate over all disk orientations and get the effective moduli $\bar{\kappa}'$ and $\bar{\mu}'$. As in the effective stiffness model, no distinction is made between finite needles (disks) and infinite fibres (layers). Table 2 shows that in the cases of spherical, needle- and disk-shaped inclusions, $\bar{\kappa}$ calculated from (4.7), (4.3), (5.1) and (5.2) are slightly higher than the exact $\bar{\kappa}$ for spherical inclusions[6] and $\bar{\kappa}'$ calculated for needle- and disk-shaped inclusions according to[13].

Figures 5 and 6 show $\bar{\kappa}$ and $\bar{\mu}$ for volume fractions of inclusions $\bar{\eta} \in (0, 1)$ in the macroscopically isotropic case. The Hashin–Shtrikman bounds are drawn in dash lines. For spherical inclusions $\bar{\kappa}$ calculated from (4.7) are seen to coincide practically with the lower bound, while the exact $\bar{\kappa}$ given by (38) in[6] yields this bound exactly. For inclusions shaped like ellipsoids of revolution, $\bar{\kappa}$ calculated from (4.5), (4.6) grow larger if α moves away from 1 at fixed $\bar{\eta}$. Hence, the more elongated or flattened are the inclusions, the higher are $\bar{\kappa}$. For the limit $\alpha \to \infty$ ($\alpha \to 0$), we obtain $\bar{\kappa}$, which exceed the upper Hashin–Shtrikman bound. This indicates that $\bar{\kappa}$ become too high if the lengths of the ellipsoid axes differ in the order of magnitude. For composite elements of such a shape, the linear approximation (2.2) of the displacement vector does not seem adequate. In Section 5 the case of very elongated and very flattened inclusions was approximated by composite elements of a different shape. As Fig. 5 suggests, $\bar{\kappa}$ calculated for needle-shaped inclusions from (5.1) are higher than those for spherical inclusions and coincide practically with $\bar{\kappa}'$ calculated according to[13]. $\bar{\kappa}$ calculated for disk-shaped inclusions from (5.2) are even higher than those for needle-shaped inclusions and coincide approximatly with $\bar{\kappa}'$ obtained by the method of[13]. Figure 6 shows that in the case of the effective shear modulus $\bar{\mu}$, the coincidence of $\bar{\mu}$ for disk-shaped inclusions obtained from (5.2) with $\bar{\mu}'$ obtained by the

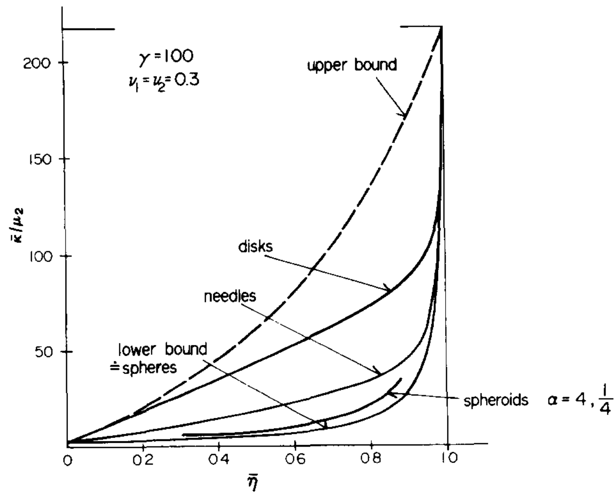

Fig. 5. Effective bulk moduli $\bar{\kappa}$.

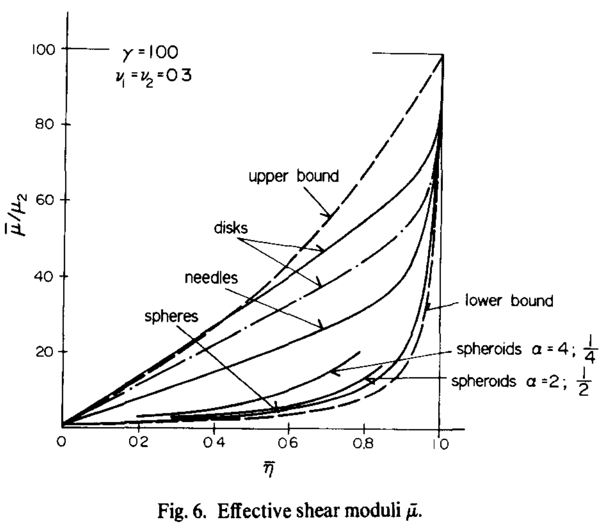

Fig. 6. Effective shear moduli $\bar{\mu}$.

method of [13] (the dot-and-dashed line in Fig. 6) is not as good as for $\bar{\kappa}$. Besides, $\bar{\mu}$ for low $\bar{\eta}$ exceed slightly the upper Hashin–Shtrikman bound.

If the inclusions are harder than the matrix, i.e.

$$\frac{\mu_1}{\mu_2} > 1, \qquad \frac{\kappa_1}{\kappa_2} > 1,$$

the self-consistent method gives for disk-shaped inclusions the effective moduli coinciding with the upper Hashin–Shtrikman bound [11]. Figures 5 and 6 show that $\bar{\kappa}$ and $\bar{\mu}$ calculated from (5.2) and by the method of [13] can be considerably lower for this case. To explain this, note that the self-consistent method [9–10] fails to give consideration to the microscopical arrangement of inclusions, while in the present model the material is built up of composite elements of a special shape. It is seen from the form of the composite disk-shaped element that the role of the inclusions and the matrix may be interchanged with $\bar{\eta}$ and γ changed correspondingly to $1-\bar{\eta}$ and $1/\gamma$. But now the matrix become harder than the inclusions and the self-consistent method yields the lower bound [11]. Thus, the results of the self-consistent method can not apply to our special microscopical arrangement of inclusions. If the inclusions are rigid the self-consistent method yields infinitely high effective moduli for arbitrarily low concentrations $\bar{\eta}$ of disk-shaped inclusions. This could be the case for such a special microscopical arrangement of inclusions if the disks formed a rigid framework. It seems that for macroscopically isotropic composites the scatter of the effective moduli due to various microscopical arrangements of disk-shaped inclusions is considerable.

7. CONCLUSIONS

The object of the study was to find approximate effective moduli of elastic composites consisting of a matrix and inclusions of various shapes. These moduli were obtained by examining the phase velocities of harmonic waves in the effective stiffness model. Aligned or randomly oriented inclusions shaped like ellipsoids, needles and disks were considered in the study. The model referred to a special microscopic arrangement of the inclusions, which enables the material to be built up of composite elements whose outer shape resembles the shape of the inclusions.

In the approximate model the displacement in the inclusion and in the matrix is linearized. The interaction between the inclusion and the matrix and between the neighbouring composite elements is taken into account by simulating point by point continuity of the displacement at the interfaces. Stress boundary conditions at the interfaces between the inclusion and the matrix can not be included in the first-order approximation used in this paper. A second-order formulation of the effective stiffness method was elaborated in [12] for laminated media. Both continuous and discontinuous stress vectors at the layer interfaces were considered and the differences in the dispersion curves were found negligible. The second-order formulation gave a better approximation to the exact dispersion curves for shorter wave lengths giving, however, the same phase velocities at infinite wave-lengths as the first-order formulation. It seems that for the composites considered in this paper a higher-order approximation to the displacement field and the stress vector continuity requirement would also have but a slight effect on the effective moduli results.

In the cases examined in Section 6, the approximate moduli ranged within the bounds of Hashin–Shtrikman (with some exceptions when the upper bound was slightly exceeded). Whenever the approximate moduli could be compared with the exact ones [6, 8], the approximate moduli were always slightly higher. The moduli were obtained explicitly, and the model is not restricted to dilute suspensions of inclusions. Variously shaped inclusions can be considered to be present simultaneously in the material, and the model can be generalized to several phases.

Considerable differences were found to exist when comparing our results obtained for disk-shaped inclusions with those of the self-consistent method, which gives the moduli coinciding with the Hashin–Shtrikman bound. The self-consistent method fails to give consideration to the microscopical arrangement of inclusions. Its disadvantage lies in the fact that a system of algebraic equations must be solved in the calculation of the moduli. The moduli are obtained explictly only in the case of disk-shaped inclusions [11].

It was shown in Section 6 that the microscopical arrangement of inclusions may affect the effective moduli considerably. A shortcoming of the present method is that it assumes only a special microscopical arrangement of inclusions. The phase geometry of real materials can be described only statistically. In this connection we mention here the interesting papers[14–15] whose authors, Bose and Mal, studied the propagation of harmonic waves in statistically uniform fibre-reinforced composites. If the correlation in the position of the fibres can be ignored, their formulae for the axial shear modulus and the transverse bulk modulus lead to the exact Hashin–Rosen expressions[8].

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APPENDIX

(1) For the case of aligned ellipsoidal inclusions, the non-zero components of the tensors $\tilde{A}_{ijkl}$, $\tilde{B}_{ijkl}$, $\tilde{C}_{ijkl}$ from (2.8) are

$$
\begin{aligned}
&\tilde{A}_{1111}=\eta^3c_{11}+(1-\eta^3)(\lambda_2+2\mu_2),\quad \tilde{A}_{2222}=\eta^3c_{22}+(1-\eta^3)(\lambda_2+2\mu_2),\\
&\tilde{A}_{3333}=\eta^3c_{33}+(1-\eta^3)(\lambda_2+2\mu_2),\quad \tilde{A}_{1122}=\tilde{A}_{2211}=\eta^3c_{12}+(1-\eta^3)\lambda_2,\\
&\tilde{A}_{1133}=\tilde{A}_{3311}=\eta^3c_{13}+(1-\eta^3)\lambda_2,\quad \tilde{A}_{2233}=\tilde{A}_{3322}=\eta^3c_{23}+(1-\eta^3)\lambda_2,\\
&\tilde{A}_{1212}=\tilde{A}_{1221}=\tilde{A}_{2112}=\tilde{A}_{2121}=\eta^3c_{66}+(1-\eta^3)\mu_2,\\
&\tilde{A}_{1313}=\tilde{A}_{1331}=\tilde{A}_{3113}=\tilde{A}_{3131}=\eta^3c_{55}+(1-\eta^3)\mu_2,\\
&\tilde{A}_{2323}=\tilde{A}_{2332}=\tilde{A}_{3223}=\tilde{A}_{3232}=\eta^3c_{44}+(1-\eta^3)\mu_2,\\
\\
&\tilde{B}_{1111}=\eta^3[(\lambda_2+2\mu_2)-c_{11}],\quad \tilde{B}_{2222}=\eta^3[(\lambda_2+2\mu_2)-c_{22}],\\
&\tilde{B}_{3333}=\eta^3[(\lambda_2+2\mu_2)-c_{33}],\quad \tilde{B}_{1122}=\tilde{B}_{2211}=\eta^3(\lambda_2-c_{12}),\\
&\tilde{B}_{1133}=\tilde{B}_{3311}=\eta^3(\lambda_2-c_{13}),\quad \tilde{B}_{2233}=\tilde{B}_{3322}=\eta^3(\lambda_2-c_{23}),\\
&\tilde{B}_{1212}=\tilde{B}_{1221}=\tilde{B}_{2112}=\tilde{B}_{2121}=\eta^3(\mu_2-c_{66}),\\
&\tilde{B}_{1313}=\tilde{B}_{1331}=\tilde{B}_{3113}=\tilde{B}_{3131}=\eta^3(\mu_2-c_{55}),\\
&\tilde{B}_{2323}=\tilde{B}_{2332}=\tilde{B}_{3223}=\tilde{B}_{3232}=\eta^3(\mu_2-c_{44}),\\
\\
&\tilde{C}_{1111}=\eta^3c_{11}+(3V-\eta^3)\lambda_2+\left[V\left(6+\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)-2\eta^3\right]\mu_2,\\
&\tilde{C}_{2222}=\eta^3c_{22}+(3V-\eta^3)\lambda_2+\left[V\left(6+\frac{b^2}{a^2}+\frac{b^2}{c^2}\right)-2\eta^3\right]\mu_2,\\
&\tilde{C}_{3333}=\eta^3c_{33}+(3V-\eta^3)\lambda_2+\left[V\left(6+\frac{c^2}{a^2}+\frac{c^2}{b^2}\right)-2\eta^3\right]\mu_2,\\
&\tilde{C}_{1122}=\tilde{C}_{2211}=\eta^3c_{12}+(V-\eta^3)\lambda_2+V\mu_2,\\
&\tilde{C}_{1133}=\tilde{C}_{3311}=\eta^3c_{13}+(V-\eta^3)\lambda_2+V\mu_2,\\
&\tilde{C}_{2233}=\tilde{C}_{3322}=\eta^3c_{23}+(V-\eta^3)\lambda_2+V\mu_2,\\
&\tilde{C}_{1212}=\eta^3c_{66}+V\frac{a^2}{b^2}\lambda_2+\left[V\left(3+2\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)-\eta^3\right]\mu_2,\\
&\tilde{C}_{2121}=\eta^3c_{66}+V\frac{b^2}{a^2}\lambda_2+\left[V\left(3+2\frac{b^2}{a^2}+\frac{b^2}{c^2}\right)-\eta^3\right]\mu_2,\\
&\tilde{C}_{1221}=\tilde{C}_{2112}=\eta^3c_{66}+V\lambda_2+(V-\eta^3)\mu_2,\\
&\tilde{C}_{1313}=\eta^3c_{55}+V\frac{a^2}{c^2}\lambda_2+\left[V\left(3+2\frac{a^2}{c^2}+\frac{a^2}{b^2}\right)-\eta^3\right]\mu_2,\\
&\tilde{C}_{3131}=\eta^3c_{55}+V\frac{c^2}{a^2}\lambda^2+\left[V\left(3+2\frac{c^2}{a^2}+\frac{c^2}{b^2}\right)-\eta^3\right]\mu_2,
\end{aligned}
\tag{A1}
$$

$$\tilde{C}_{1331}=\tilde{C}_{3113}=\eta^3c_{55}+V\lambda_2+(V-\eta^3)\mu_2,$$
$$\tilde{C}_{2323}=\eta^3c_{44}+V\frac{b^2}{a^2}\lambda_2+\left[V\left(3+2\frac{b^2}{c^2}+\frac{b^2}{a^2}\right)-\eta^3\right]\mu_2,$$
$$\tilde{C}_{3232}=\eta^3c_{44}+V\frac{c^2}{b^2}\lambda_2+\left[V\left(3+2\frac{c^2}{b^2}+\frac{c^2}{a^2}\right)-\eta^3\right]\mu_2,$$
$$\tilde{C}_{2332}=\tilde{C}_{3223}=\eta^3c_{44}+V\lambda_2+(V-\eta^3)\mu_2,$$

where

$$\eta=\frac{r_1}{r_2},\qquad V=\frac{\eta^2}{5(1-\eta)}.$$

By comparing the phase velocities of the corresponding waves propagating in the directions $x_i (i=1,2,3)$ for the composite material and for a homogeneous orthotropic material with the material constants (2.16) and the mass density $\bar{\rho}$ defined by (2.11), the expressions for the moduli $\bar{c}_{11}$, $\bar{c}_{22}$, $\bar{c}_{33}$, $\bar{c}_{44}$, $\bar{c}_{55}$ and $\bar{c}_{66}$ are obtained. For example, $\bar{c}_{11}$, $\bar{c}_{22}$ and $\bar{c}_{66}$ are of the form

$$\bar{c}_{11}=\frac{1}{C}\begin{vmatrix}a_1&a_4&a_5&a_{16}\\a_4&a_{14}&a_{15}&a_{16}\\a_5&a_{15}&a_{17}&a_{18}\\a_6&a_{16}&a_{18}&a_{19}\end{vmatrix},\quad \bar{c}_{22}=\frac{1}{C}\begin{vmatrix}a_9&a_5&a_{11}&a_{12}\\a_5&a_{14}&a_{15}&a_{16}\\a_{11}&a_{15}&a_{17}&a_{18}\\a_{12}&a_{16}&a_{18}&a_{19}\end{vmatrix},$$

$$\bar{c}_{66}=\frac{1}{C'}\begin{vmatrix}a_{10}&a_{13}&a_7\\a_{13}&a_{20}&a_{21}\\a_7&a_{21}&a_{22}\end{vmatrix}=\frac{1}{C'}\begin{vmatrix}a_2&a_7&a_8\\a_7&a_{20}&a_{21}\\a_8&a_{21}&a_{22}\end{vmatrix},$$

$$C=\begin{vmatrix}a_{14}&a_{15}&a_{16}\\a_{15}&a_{17}&a_{18}\\a_{16}&a_{18}&a_{19}\end{vmatrix},\quad C'=\begin{vmatrix}a_{20}&a_{21}\\a_{21}&a_{22}\end{vmatrix}. \tag{A2}$$

Here $a_1, a_2, \ldots, a_{22}$ are of the form

$$\begin{aligned}
&a_1=\tilde{A}_{1111}+2\tilde{B}_{1111}+\tilde{C}_{1111},\quad a_2=\tilde{A}_{1212}+2\tilde{B}_{1212}+\tilde{C}_{2121},\\
&a_3=\tilde{A}_{1122}+2\tilde{B}_{1122}+\tilde{C}_{1122}+\tilde{A}_{1212}+2\tilde{B}_{1212}+\tilde{C}_{1221},\\
&a_4=\tilde{B}_{1111}+\tilde{C}_{1111},\quad a_5=\tilde{B}_{1122}+\tilde{C}_{1122},\\
&a_6=\tilde{B}_{1133}+\tilde{C}_{1133},\quad a_7=\tilde{B}_{1212}+\tilde{C}_{1221},\\
&a_8=\tilde{B}_{1212}+\tilde{C}_{2121},\quad a_9=\tilde{A}_{2222}+2\tilde{B}_{2222}+\tilde{C}_{2222},\\
&a_{10}=\tilde{A}_{1212}+2\tilde{B}_{1212}+\tilde{C}_{1212},\quad a_{11}=\tilde{B}_{2222}+\tilde{C}_{2222},\\
&a_{12}=\tilde{B}_{2233}+\tilde{C}_{2233},\quad a_{13}=\tilde{B}_{1212}+\tilde{C}_{1212},\\
&a_{14}=\tilde{C}_{1111},\quad a_{15}=\tilde{C}_{1122},\quad a_{16}=\tilde{C}_{1133},\\
&a_{17}=\tilde{C}_{2222},\quad a_{18}=\tilde{C}_{2233},\quad a_{19}=\tilde{C}_{3333},\\
&a_{20}=\tilde{C}_{1212},\quad a_{21}=\tilde{C}_{1221},\quad a_{22}=\tilde{C}_{2121}.
\end{aligned} \tag{A3}$$

To obtain $\bar{c}_{12}$ we examine the wave (2.15) for $n_3=0$, $n_1\neq 0$, $n_2\neq 0$. The system (2.13) decomposes into two systems, the first containing seven equations for U_1, U_2, Ψ_{11}, Ψ_{22}, Ψ_{33}, Ψ_{12} and Ψ_{21}, the second the equations for U_3, Ψ_{13}, Ψ_{31}, Ψ_{23} and Ψ_{32}. By comparing the phase velocities c in the condition of non-zero solution to the first system of equations

$$\begin{vmatrix}
a_1n_1^2+a_2n_2^2-\bar{\rho}c^2 & a_4n_1 & a_5n_1 & a_6n_1 & a_3n_1n_2 & a_7n_2 & a_8n_2\\
a_4n_1 & a_{14} & a_{15} & a_{16} & a_5n_2 & 0 & 0\\
a_5n_1 & a_{15} & a_{17} & a_{18} & a_{11}n_2 & 0 & 0\\
a_6n_1 & a_{16} & a_{18} & a_{19} & a_{12}n_2 & 0 & 0\\
a_3n_1n_2 & a_5n_2 & a_{11}n_2 & a_{18}n_2 & a_9n_2^2+a_{10}n_1^2-\bar{\rho}c^2 & a_{13}n_1 & a_7n_1\\
a_7n_2 & 0 & 0 & 0 & a_{13}n_1 & a_{20} & a_{21}\\
a_8n_2 & 0 & 0 & 0 & a_7n_1 & a_{21} & a_{22}
\end{vmatrix}=0$$

with the corresponding condition for the homogeneous material

$$\begin{vmatrix}
\bar{c}_{11}n_1^2+\bar{c}_{66}n_2^2-\bar{\rho}c^2 & (\bar{c}_{12}+\bar{c}_{66})n_1n_2\\
(\bar{c}_{12}+\bar{c}_{66})n_1n_2 & \bar{c}_{66}n_1^2+\bar{c}_{22}n_2^2-\bar{\rho}c^2
\end{vmatrix}=0$$

and using (A2), we obtain

$$\bar{c}_{12}+\bar{c}_{66}=\frac{1}{C'}\begin{vmatrix}a_3&a_7&a_8\\a_{13}&a_{20}&a_{21}\\a_7&a_{21}&a_{22}\end{vmatrix}+\frac{1}{C}\begin{vmatrix}0&a_5&a_{11}&a_{12}\\a_4&a_{14}&a_{15}&a_{16}\\a_5&a_{15}&a_{17}&a_{18}\\a_6&a_{16}&a_{18}&a_{19}\end{vmatrix} \tag{A4}$$

The same procedure applied to the wave (2.15) for $n_2=0$, $n_1\neq 0$, $n_3\neq 0$, and for $n_1=0$, $n_2\neq 0$, $n_3\neq 0$ would result in similar expressions for $\bar{c}_{13}+\bar{c}_{55}$ and $\bar{c}_{23}+\bar{c}_{44}$. Then, after some manipulations, (A2)–(A4) yield the effective moduli in the form of (2.17).

(2) From the local basis $\tilde{x}_i$ one can pass to the global basis x_i generally by the sequence of three rotations $T^{(1)}_{ij}$, $T^{(2)}_{ij}$, $T^{(3)}_{ij}$ defined by

$$x_i = T^{(1)}_{ij}x''_j, \quad x''_i = T^{(2)}_{ij}x'_j, \quad x'_i = T^{(3)}_{ij}\tilde{x}_j,$$

$$T^{(1)}_{ij}(\varphi) = \begin{Vmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \quad T^{(2)}_{ij}(\vartheta) = \begin{Vmatrix} \cos\vartheta & 0 & \sin\vartheta \\ 0 & 1 & 0 \\ -\sin\vartheta & 0 & \cos\vartheta \end{Vmatrix}, \quad T^{(3)}_{ij}(\omega) = \begin{Vmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{A5}$$

$$\varphi \in \langle 0, 2\pi \rangle, \quad \vartheta \in \left\langle 0, \frac{\pi}{2} \right\rangle, \quad \omega \in \langle 0, \pi \rangle.$$

The basis x_i passes to the basis x''_i by rotating through an angle φ about $x_3 \equiv x''_3$ in the positive direction, the basis x''_i passes to the basis x'_i by rotating through an angle ϑ about $x''_2 \equiv x'_2$ in the negative direction and the basis x'_i passes to the basis $\tilde{x}_i$ by rotating through an angle ω about $x'_3 \equiv \tilde{x}_3$ again in the positive direction. Hence the transformation of T_{ij} from $\tilde{x}_i$ to x_i

$$x_i = T_{ij}\tilde{x}_j, \quad T_{ij}(\varphi, \vartheta, \omega) = T^{(1)}_{ik}(\varphi)T^{(2)}_{kl}(\vartheta)T^{(3)}_{lj}(\omega) \tag{A6}$$

depends on three independent Euler's angles φ, ϑ, ω. For $\omega \equiv 0$, $T^{(3)}_{ij}$ is a unit matrix, and T_{ij} takes the form

$$T_{ij}(\varphi, \vartheta) = T^{(1)}_{ik}(\varphi)T^{(2)}_{kj}(\vartheta). \tag{A7}$$

(3) Isotropically distributed inclusions in the shape of ellipsoids of revolution. After substituting (4.1) into (A1) and carrying out the integration in (4.2), the non-zero components of A_{ijkl}, B_{ijkl}, C_{ijkl} defining $\mathcal{W}$ in (4.2) become

$$\begin{aligned}
&A_{1111} = A_{2222} = A_{3333} = \frac{\eta^3}{15}(8c_{11} + 3c_{33} + 4c_{13} + 8c_{44}) + (1-\eta^3)(\lambda_2 + 2\mu_2), \\
&A_{1122} = A_{2211} = A_{1133} = A_{3311} = A_{2233} = A_{3322} \\
&\quad = \frac{\eta^3}{15}(c_{11} + c_{33} + 5c_{12} + 8c_{13} - 4c_{44}) + (1-\eta^3)\lambda_2, \\
&A_{1212} = A_{2112} = A_{1221} = A_{2121} = A_{1313} = A_{3113} \\
&\quad = A_{1331} = A_{3131} = A_{2323} = A_{3223} = A_{2332} \\
&\quad = A_{3232} = \frac{\eta^3}{30}(7c_{11} + 2c_{33} - 5c_{12} - 4c_{13} + 12c_{44}) + (1-\eta^3)\mu_2,
\end{aligned}$$

$$\begin{aligned}
&B_{1111} = B_{2222} = B_{3333} = \eta^3\left[(\lambda_2 + 2\mu_2) - \frac{1}{15}(8c_{11} + 3c_{33} + 4c_{13} + 8c_{44})\right], \\
&B_{1122} = B_{2211} = B_{1133} = B_{3311} = B_{2233} = B_{3322} \\
&\quad = \eta^3\left[\lambda_2 - \frac{1}{15}(c_{11} + c_{33} + 5c_{12} + 8c_{13} - 4c_{44})\right], \\
&B_{1212} = B_{2112} = B_{1221} = B_{2121} = B_{1313} = B_{3113} \\
&\quad = B_{1331} = B_{3131} = B_{2323} = B_{3223} = B_{2332} \\
&\quad = B_{3232} = \eta^3\left[\mu_2 - \frac{1}{30}(7c_{11} + 2c_{33} - 5c_{12} - 4c_{13} + 12c_{44})\right],
\end{aligned} \tag{A8}$$

$$\begin{aligned}
&C_{1111} = C_{2222} = C_{3333} = \frac{\eta^3}{15}(8c_{11} + 3c_{33} + 4c_{13} + 8c_{44}) \\
&\quad + \left\langle \frac{1}{15}\left[41 + 2\left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V - \eta^3 \right\rangle \lambda_2 + \left\langle \frac{4}{5}\left[8 + \left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V - 2\eta^3 \right\rangle \mu_2, \\
&C_{1122} = C_{2211} = C_{1133} = C_{3311} = C_{2233} = C_{3322} = \frac{\eta^3}{15}(c_{11} + c_{33} + 5c_{12} + 8c_{13} - 4c_{44}) \\
&\quad + \left\langle \frac{1}{15}\left[17 - \left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V - \eta^3 \right\rangle \lambda_2 + \frac{1}{15}\left[17 - \left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V\mu_2, \\
&C_{1212} = C_{2121} = C_{1313} = C_{3131} = C_{2323} = C_{3232} = \frac{\eta^3}{30}(7c_{11} + 2c_{33} - 5c_{12} - 4c_{13} + 12c_{44}) \\
&\quad + \frac{1}{15}\left[7 + 4\left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V\lambda_2 + \left\langle \frac{2}{15}\left[31 + 7\left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V - \eta^3 \right\rangle \mu_2, \\
&C_{1221} = C_{2112} = C_{1331} = C_{3113} = C_{2332} = C_{3223} = \frac{\eta^3}{30}(7c_{11} + 2c_{33} - 5c_{12} - 4c_{13} + 12c_{44}) \\
&\quad + \frac{1}{15}\left[17 - \left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V\lambda_2 + \left\langle \frac{1}{15}\left[17 - \left(\alpha^2 + \frac{1}{\alpha^2}\right)\right]V - \eta^3 \right\rangle \mu_2.
\end{aligned}$$

V was defined in (A1).
$b_1, b_2, \ldots, b_{10}$ in (4.3) are of the form

$$b_1 = A_{1111} + 2B_{1111} + C_{1111},$$

$$b_2 = B_{1111} + C_{1111}, \quad b_3 = B_{1122} + C_{1122},$$
$$b_4 = C_{1111}, \quad b_5 = C_{1122},$$
$$b_6 = A_{1212} + 2B_{1212} + C_{1212},$$
$$b_7 = B_{1212} + C_{1212}, \quad b_8 = B_{1212} + C_{1221},$$
$$b_9 = C_{1212}, \quad b_{10} = C_{1221},$$

where A_{ijkl}, B_{ijkl}, C_{ijkl} are given in (A8).